

THE FREE PRODUCT OF MATROIDS

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ABSTRACT. We introduce a noncommutative binary operation on matroids, called free product. We show that this operation respects matroid duality, and has the property that, given only the cardinalities, an ordered pair of matroids may be recovered, up to isomorphism, from its free product. We use these results to give a short proof of Welsh's 1969 conjecture, which provides a progressive lower bound for the number of isomorphism classes of matroids on an n -element set.

In the *minor coalgebra* of matroids ([3], [1]), the coproduct of a matroid $M(S)$ is given by $\sum_{A \subseteq S} M|A \otimes M/A$, where $M|A$ denotes the restriction of M to A and M/A denotes the matroid on the set difference $S \setminus A$ obtained by contracting A from M . The product of matroids M and N in the dual algebra is thus a linear combination $\sum_L \alpha_L L$ of those matroids L having some restriction isomorphic to M , with complementary contraction isomorphic to N . The coefficient α_L of $L = L(U)$ is the number of subsets $A \subseteq U$ such that $L|A \cong M$ and $L/A \cong N$. If the matroids having nonzero coefficient in the product of M and N are ordered in the weak-map order, there is a final term equal to a scalar multiple of the direct product $M \oplus N$, and an initial term equal to a scalar multiple of a matroid that we have elected to call the *free product* of M and N .

In the present short article we give an intrinsic definition of the free product of matroids, and prove the crucial result that, given only their cardinalities, the two factors themselves, and even the order of the factors, can be recovered, up to isomorphism, from the free product. This is in sharp contrast to the behavior of direct sums, where the failure of unique ordered factorization gave rise to a little crisis in matroid theory, holding up the proof of Welsh's "self-evident" conjecture [4] for more than three decades. He conjectured that if there are f_n isomorphism classes of matroids on an n -element set, then $f_n \cdot f_m \leq f_{n+m}$, for all $n, m \geq 0$. Where direct sum fails, free product succeeds; we prove the conjecture here.

In future work we shall investigate in detail the combinatorial properties of the free product, as well as its implications for the minor coalgebra of matroids.

We denote the rank and nullity functions of a matroid $M(S)$ by ρ_M and ν_M , respectively, and denote by λ_M the *rank-lack* function on M , given by $\lambda_M(A) = \rho(M) - \rho_M(A)$, for all $A \subseteq S$, where $\rho(M) = \rho_M(S)$ is the rank of M . We denote the disjoint union of sets S and T by $S + T$ and the intersection $S \cap T$ by either S_T or T_S . We refer the reader to Oxley's book [2] for any background on matroid theory that might be needed.

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Proposition 1. *For all matroids $M(S)$ and $N(T)$, the collection*

$$\mathcal{I} = \{A \subseteq S + T : A_S \text{ is independent in } M \text{ and } \lambda_M(A_S) \geq \nu_N(A_T)\}$$

is the family of independent subsets of a matroid $M \square N$ on $S + T$.

Proof. Suppose that $A \in \mathcal{I}$ and $B \subseteq A$. Then $B_S \subseteq A_S$ and $B_T \subseteq A_T$, and so $\lambda_M(B_S) \geq \lambda_M(A_S)$ and $\nu_N(A_T) \geq \nu_N(B_T)$. Since $A \in \mathcal{I}$, the set A_S is independent in M and $\lambda_M(A_S) \geq \nu_N(A_T)$, from which it follows that B_S is independent in M and $\lambda_M(B_S) \geq \nu_N(B_T)$, that is, $B \in \mathcal{I}$.

Now suppose that $A, B \in \mathcal{I}$, with $|A| < |B|$. We consider three cases: First, if $\lambda_M(A_S) > \nu_N(A_T)$ and $|A_S| < |B_S|$, then $\lambda_M(A_S \cup x) \geq \nu_N(A_T)$, for any $x \in B_S \setminus A_S$, so if we choose any such x with $A_S \cup x$ independent in M , then $A \cup x = (A_S \cup x) \cup A_T$ belongs to \mathcal{I} . Second, if $\lambda_M(A_S) > \nu_N(A_T)$ and $|A_T| < |B_T|$, then $\lambda_M(A_S) \geq \nu_N(A_T \cup y)$, for any $y \in B_T \setminus A_T$, and so $A \cup y = A_S \cup (A_T \cup y) \in \mathcal{I}$. Finally, we consider the case in which $\lambda_M(A_S) = \nu_N(A_T)$. Since $A \in \mathcal{I}$, the set A_S is independent in M , so that $\rho_M(A_S) = |A_S|$, and hence in this case, $\lambda_M(A_S) = \rho(M) - |A_S| = \nu_N(A_T)$. Also, since $B \in \mathcal{I}$, we have $\lambda_M(B_S) = \rho(M) - |B_S| \geq \nu_N(B_T)$, and thus

$$\begin{aligned} \rho_N(A_T) &= |A_T| - \nu_N(A_T) \\ &= |A_T| + |A_S| - \rho(M) \\ &< |B_T| + |B_S| - \rho(M) \\ &\leq |B_T| - \nu_N(B_T) \\ &= \rho_N(B_T). \end{aligned}$$

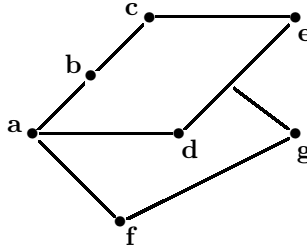
Therefore we may find an element z of $B_T \setminus A_T$ such that $\nu_N(A_T \cup z) = \nu_N(A_T)$, and so $A \cup z = A_S \cup (A_T \cup z)$ belongs to \mathcal{I} . Hence $M \square N$ is a matroid. \square

We refer to the matroid $M \square N$ as the *free product* of M and N . Note that the set of bases of $M(S) \square N(T)$ is given by

$$\{A \subseteq S + T : A_S \text{ is independent in } M, A_T \text{ spans } N, \text{ and } \lambda_M(A_S) = \nu_N(A_T)\},$$

and so, in particular, $\rho(M \square N) = \rho(M) + \rho(N)$, for all M and N .

Example 2. Let $S = \{a, b, c\}$ and $T = \{d, e, f, g\}$, and suppose that $M(S)$ is a three-point line and N consists of two double points, de and fg . The free product $M \square N$ is shown below:



The bases of $M \square N$ are the sets of the form $A \cup B$, with $A \subseteq S$, $B \subseteq T$, and either

- (i) $A = \emptyset$ and $B = T$,
- (ii) $|A| = 1$ and $|B| = 3$, or
- (iii) $|A| = 2$ and $|B| = 2$, with B not equal to $\{d, e\}$ or $\{f, g\}$.

The following proposition verifies that, in a free product $M(S) \sqcup N(T)$, the restriction to S and contraction by S yield M and N as minors.

Proposition 3. *For all matroids $M(S)$ and $N(T)$,*

$$(M \sqcup N)|_S = M \quad \text{and} \quad (M \sqcup N)/S = N.$$

Proof. It is immediate from the definition of independence in $L = M \sqcup N$ that $A \subseteq S$ is independent in L if and only if it is independent in M , and hence $L|_S = M$. Let $E \subseteq S$ be a basis of M . The independent sets of L/S are those sets $B \subseteq T$ such that $E \cup B$ is independent in L . Since $\lambda_M(E) = 0$, the set $E \cup B$ is independent in L if and only if B is independent in N , and hence $L/S = N$. \square

For any matroid $M = M(S)$, the rank function of the dual matroid M^* satisfies $\rho_{M^*}(B) = |B| - \rho(M) + \rho_M(A)$, or equivalently, $\lambda_M(A) = \nu_{M^*}(B)$, for all $A + B = S$.

Proposition 4. *For all matroids M and N , $(M \sqcup N)^* = N^* \sqcup M^*$.*

Proof. Suppose that $M = M(S)$, $N = N(T)$, and $A + B = S + T$, so that A is a basis for $M \sqcup N$ if and only if B is a basis for $(M \sqcup N)^*$. Now A is a basis for $M \sqcup N$ if and only if A_S is independent in M , A_T spans N and $\lambda_M(A_S) = \nu_N(A_T)$, which is true if and only if B_S spans M^* , B_T is independent in N^* , and $\nu_{M^*}(B_S) = \lambda_{N^*}(B_T)$, that is, if and only if B is a basis for $N^* \sqcup M^*$. \square

The next result implies that, given the size of S , we can recover the rank of $M(S)$ from the free product $M \sqcup N$.

Lemma 5. *If $L = M(S) \sqcup N(T)$, then $\rho(M) \leq \rho_L(U)$, for all $U \subseteq S + T$ such that $|U| = |S|$.*

Proof. Suppose that $U \subseteq S + T$, with $|U| = |S|$, and let V be the complement of U in $S + T$. Note that V_S is the complement of U_S in S , while U_T is the complement of U_S in U , so that $|V_S| = |U_T|$. Let $E \subseteq U_S$ be a basis for $M|_{U_S} = L|_{U_S}$. Since $\lambda_M(U_S) \leq |V_S| = |U_T|$, there exists $A \subseteq U_T$ with $|A| = \lambda_M(U_S)$. It follows that $E \cup A$ is independent in L , and thus, since $E \cup A \subseteq U$, we have $\rho_L(U) \geq |E| + |A| = \rho_M(U_S) + \lambda_M(U_S) = \rho(M)$. \square

For any matroid M , we write $\text{Loop}(M)$ and $\text{Isth}(M)$, respectively, for the sets of loops and isthmuses of M . The following result shows, in particular, that whenever M is isthmusless and N loopless, and we know the size of M , then the support set of M , and thus the matroid M itself, can be recovered from the free product $M \sqcup N$.

Lemma 6. *Suppose that $L = M(S) \sqcup N(T)$, and $U \subseteq S + T$ satisfies $|U| = |S|$. If U contains a nonloop of N and the complement of U in $S + T$ contains a nonisthmus of M , then $\rho_L(U) > \rho(M)$.*

Proof. Let V denote the complement of U in $S + T$. Since $V_S \not\subseteq \text{Isth}(M)$, and $|U| = |S|$, it follows that $\lambda_M(U_S) < |V_S| = |U_T|$. If F is any subset of U_T containing at least one nonloop and having size $\lambda_M(U_S) + 1$, then $\lambda_M(U_S) \geq \nu_N(F)$, and thus $E \cup F$ is independent in L , for all independent E in $M|_{U_S}$. In particular, if E is a basis for $M|_{U_S}$, then $|E \cup F| = |E| + |F| > \rho_M(U_S) + \lambda_M(U_S) = \rho(M)$, and hence $\rho_L(U) > \rho(M)$. \square

Our main result, below, implies that, given the size of M , we may recover M and N up to isomorphism from $M \sqcup N$, without conditions on M and N .

Theorem 7. *If $L = M(S) \square N(T)$, then for any $U \subseteq S + T$ such that $|U| = |S|$ and $\rho_L(U) = \rho(M)$, there exist bijective weak maps $L|U \rightarrow M$ and $L/U \rightarrow N$.*

Proof. Let V denote the complement of U in $S + T$, let $f: V_S \rightarrow U_T$ be an arbitrary bijection, and define $\phi: S + T \rightarrow S + T$ by

$$\phi(x) = \begin{cases} f(x), & \text{if } x \in V_S, \\ f^{-1}(x), & \text{if } x \in U_T, \\ x, & \text{if } x \in U_S \cup V_T. \end{cases}$$

Denote by ϕ_1 and ϕ_2 , respectively, the restrictions $\phi|U$ and $\phi|V$, and note that $\phi_1: U \rightarrow S$ and $\phi_2: V \rightarrow T$ are bijections. We now show that ϕ_1 and ϕ_2 are the desired weak maps.

According to Lemma 6, the fact that $\rho_L(U) = \rho(M)$, implies that either $V_S \subseteq \text{Isth}(M)$ or $U_T \subseteq \text{Loop}(N)$, or both; we first consider the case in which $V_S \subseteq \text{Isth}(M)$.

Since $V_S \subseteq \text{Isth}(M)$, the bases for M are the sets of the form $A \cup V_S$, where A is a basis for $M|U_S$. Now if A is independent in $M|U_S$, then

$$\lambda_M(A) \geq |V_S| = |U_T| \geq \nu_N(U_T),$$

so that $A \cup U_T$ is independent in L , and thus the bases of $L|U$ are the sets of the form $A \cup U_T$, where A is a basis of $M|U_S$. Hence $B = A \cup V_S$ is a basis of M if and only if $\phi^{-1}(B) = A \cup U_T$ is a basis of $L|U$, and therefore ϕ_1 is an isomorphism from $L|U$ onto M .

Let A be a basis for $M|U_S$, so that $B = A \cup U_T$ is a basis for $L|U$, as seen above, and let $E = E_U \cup E_V$ be a basis for N . In order to see that $\phi_2: L/U \rightarrow N$ is a weak map, we need to show that $\phi^{-1}(E)$ is independent in L/U , or equivalently, that $\phi^{-1}(E) \cup B$ is independent in L . Now $\phi^{-1}(E) = \phi^{-1}(E_U) \cup E_V$, and thus $\phi^{-1}(E) \cup B = (\phi^{-1}(E_U) \cup A) \cup (E_V \cup U_T)$ is independent in L if and only if $\nu_N(E_V \cup U_T) \leq \lambda_M(\phi^{-1}(E_U) \cup A)$. Since $\phi^{-1}(E_U) \subseteq V_S \subseteq \text{Isth}(M)$, and A is a basis for $M|U_S$, we have

$$\begin{aligned} \lambda_M(\phi^{-1}(E_U) \cup A) &= |V_S| - |\phi^{-1}(E_U)| \\ &= |V_S| - |E_U|. \end{aligned}$$

On the other hand, since $E \subseteq E_V \cup U_T$, and E is a basis for N , we have

$$\begin{aligned} \nu_N(E_V \cup U_T) &= |E_V \cup U_T| - \rho_N(E_V \cup U_T) \\ &= |E_V| + |U_T| - |E| \\ &= |U_T| - |E_U|. \end{aligned}$$

Hence $\nu_N(E_V \cup U_T) = \lambda_M(\phi^{-1}(E_U) \cup A)$, and so $\phi^{-1}(E) \cup B$ is independent in (and in fact is a basis for) L . Thus, in the case that $V_S \subseteq \text{Isth}(M)$, we have that $\phi_1: L|U \rightarrow M$ and $\phi_2: L/U \rightarrow N$ are weak maps.

Now suppose that $U_T \subseteq \text{Loop}(N) = \text{Isth}(N^*)$. By Proposition 4, we have $L^* = N^* \square M^*$, and since $|U| = |S|$ and $\rho_L(U) = \rho(M)$, it follows that $|V| = |T|$ and $\rho_{L^*}(V) = \rho(N^*)$. Interchanging the roles of M , N , and U , respectively, with those of N^* , M^* and V in the above, we obtain that $\phi_2: L^*|V = (L/U)^* \rightarrow N^*$ and $\phi_1: L^*/V = (L|U)^* \rightarrow M^*$ are weak maps, and since $\rho((L/U)^*) = \rho(N^*)$ and $\rho((L|U)^*) = \rho(M^*)$, this implies that $\phi_1: L|U \rightarrow M$ and $\phi_2: L/U \rightarrow N$ are weak maps. \square

Corollary 8. *If $M(S) \sqcup N(T) \cong P(U) \sqcup Q(V)$, where $|S| = |U|$, then $M \cong P$ and $N \cong Q$.*

Proof. Choosing an isomorphism from $M \sqcup N$ to $P \sqcup Q$, and relabelling if necessary, we can assume that $M \sqcup N = P \sqcup Q$. Since $\rho(M) = \rho_{P \sqcup Q}(S)$ and $\rho(P) = \rho_{M \sqcup N}(U)$, it follows from Lemma 5 that $\rho(M) = \rho(P)$. We may thus apply Theorem 7 to obtain bijective weak maps $M \rightarrow P$, $N \rightarrow Q$, $P \rightarrow M$ and $Q \rightarrow N$; hence $M \cong P$ and $N \cong Q$. \square

We thus obtain the following result, which was conjectured by Welsh in [4]:

Corollary 9. *If f_n denotes the number of nonisomorphic matroids on a set of size n , then $f_{n+m} \geq f_n \cdot f_m$, for all $n, m \geq 0$.*

Proof. Denote by $\mathcal{M}(n)$ be the set of all isomorphism classes of matroids on a set of n elements, so that $|\mathcal{M}(n)| = f_n$, for all $n \geq 0$, and write $[M]$ for the isomorphism class of a matroid M . Corollary 8 says precisely that, for all $n, m \geq 0$, the map $\mathcal{M}(n) \times \mathcal{M}(m) \rightarrow \mathcal{M}(n+m)$ given by $([M], [N]) \mapsto [M \sqcup N]$ is injective, from which the inequality follows. \square

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